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On the Mode Correspondence Between Circular and Square Multimode Tapered Waveguides

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Abstract—**In an axially straight multimode circular waveguide taper excited with a pure $TE_{11}^{\bigcirc}$ dominant mode, the first and only converted mode at and near cutoff is the $TM_{11}^{\bigcirc}$ mode. It is shown that in an axially straight multimode square waveguide taper excited with a pure $TE_{10}^{\square}$ dominant mode, the $TM_{12}^{\square}$ mode corresponding to the $TM_{11}^{\bigcirc}$ mode in circular case is not the only first converted mode at and near cutoff.**

The overall behavior or coupling mechanism of waveguides is similar whether the waveguide is rectangular, square, circular, or elliptical: i.e., the overall coupling coefficient at cutoff of a converted mode or modes approaches an infinity of the order $0^{-1/4}$.

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In a previous paper[1] it was shown that for the case of two-mode weak coupling the coefficient of coupling between the $TE_{11}^{\bigcirc}$ dominant mode and the $TM_{11}^{\bigcirc}$ mode in tapered circular waveguides tends to approach an infinity of the order $0^{-1/4}$ at cutoff frequency whereas the corresponding coefficient of coupling between the $TE_{10}^{\square}$ dominant mode and the $TM_{12}^{\square}$ mode in tapered square waveguides approaches instead a zero of the order $0^{1/4}$ at cutoff frequency. No convincing physical interpretation was given for such surprisingly drastically different coupling behaviors at cutoff frequency. It is the attempt of this paper to offer a convincing explanation.

For modes adjacent to the dominant mode, the mode correspondence between circular and square waveguides can be easily identified. As the mode order goes higher the identification of corresponding modes becomes more and more difficult or impossible. The following mode arrangement shows the first six mode correspondences between the circular and square waveguides

$$\begin{array}{cccccc} \mathrm{TE}_{11}^{\circ} & \mathrm{TM}_{01}^{\circ} & \mathrm{TE}_{21}^{\circ} & \overline{\mathrm{TE}_{01}^{\circ} & \mathrm{TM}_{11}^{\circ}} & \mathrm{TE}_{31}^{\circ} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathrm{TE}_{10}^{\square} & \underline{\mathrm{TM}_{11}^{\square} & \mathrm{TE}_{11}^{\square}} & \mathrm{TE}_{02}^{\square} & \underline{\mathrm{TM}_{12}^{\square} & \mathrm{TE}_{12}^{\square}} \\ & & & + & & \\ & & & \mathrm{TE}_{20}^{\square} & & \end{array}.$$

The horizontal brackets indicate that the two modes in each bracket are degenerate in the sense that they have the same propagation constant.

In an axially straight multimode circular waveguide taper excited with a pure TE_{11}° dominant mode, the first and only converted mode at and near cutoff is the TM_{11}° mode. Although TE_{01}° and TM_{11}° are degenerate, it can be shown that the conversion from TE_{11}° mode to TE_{01}° mode is not possible in this case. In an axially straight square taper excited with a pure $\mathrm{TE}_{10}^{\square}$ dominant mode, it will be shown that the $\mathrm{TM}_{12}^{\square}$ mode corresponding to the TM_{11}° mode in circular waveguide is not the only first converted mode at and near cutoff since $\mathrm{TE}_{10}^{\square}$ mode will be partly converted into $\mathrm{TE}_{12}^{\square}$ simultaneously with $\mathrm{TM}_{12}^{\square}$. In other words the circular taper can be treated as a two-mode weak coupling case, whereas the square taper must be treated as a three-mode weak coupling case for rigorous presentation. It is in this sense that the one-to-one correspondence between the circular and square waveguides breaks down, when the guides are tapered at and near cutoff.

To show that three-mode coupling is the case for rectangular or square tapers excited by pure $\mathrm{TE}_{10}^{\square}$ mode with conversion into both $\mathrm{TM}_{12}^{\square}$ and $\mathrm{TE}_{12}^{\square}$ modes simultaneously at and near cutoff, we shall prove that there is coupling between $\mathrm{TE}_{10}^{\square}$ and $\mathrm{TE}_{12}^{\square}$ modes. The field configuration of the three modes at any rectangular or square cross section, as shown in Fig. 1, can be represented by

$$H_z = \sum_{q=0}^{2} C_{1q} \frac{a}{\pi} \frac{k_{1q}^2}{j\omega\mu} V_{1q} \sin\frac{\pi x}{a} \cos\frac{q\pi y}{b} \qquad q \text{ even only.}$$

$$H_y = \frac{2a}{b} C_{12} I_{12} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} - \frac{b}{2a} \bar{C}_{12} \bar{I}_{12} \sin\frac{\pi x}{a} \cdot \sin\frac{2\pi y}{b}$$

$$H_x = -\sum_{q=0}^{2} C_{1q} I_{1q} \cos\frac{\pi x}{a} \cos\frac{q\pi y}{b} - \bar{C}_{12}\bar{I}_{12} \cos\frac{\pi x}{a} \cdot \cos\frac{2\pi y}{b}$$

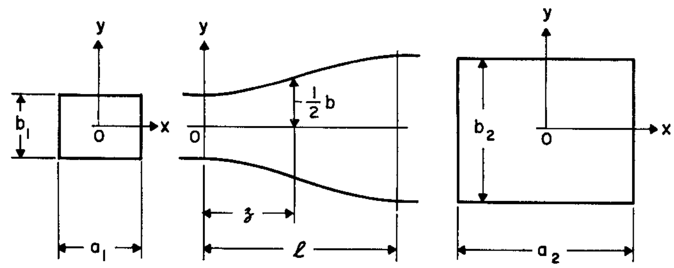

Fig. 1. A rectangular waveguide taper.

$$E_z = -\frac{b}{2\pi} \bar{C}_{12} \frac{\bar{k}_{12}^2}{j\omega\epsilon} \bar{I}_{12} \cos\frac{\pi x}{a} \sin\frac{2\pi y}{b}$$

$$E_y = \sum_{q=0}^{2} C_{1q} V_{1q} \cos\frac{\pi x}{a} \cos\frac{q\pi y}{b} + \bar{C}_{12}\bar{V}_{12} \cos\frac{\pi x}{a} \cdot \cos\frac{2\pi y}{b}$$

$$E_x = \frac{2a}{b} C_{12} V_{12} \sin\frac{\pi_1 x}{a} \sin\frac{2\pi y}{b} - \frac{b}{2a} \bar{C}_{12} \bar{V}_{12} \sin\frac{\pi x}{a} \cdot \sin\frac{2\pi y}{b} \qquad (1)$$

where barred letters denote quantities related to the TM mode,

$$C_{10} = \frac{\pi}{a} \frac{1}{k_{10}} \sqrt{\frac{2}{ab}}, \qquad C_{12} = \frac{\pi}{a} \frac{1}{k_{12}} \frac{2}{\sqrt{ab}};$$

$$\bar{C}_{12} = \frac{2\pi}{b} \frac{1}{\bar{k}_{12}} \frac{2}{\sqrt{ab}},$$

and

$$k_{pq}(z) = \left[\left(\frac{p\pi}{a(z)}\right)^2 + \left(\frac{q\pi}{q(z)}\right)^2\right]^{1/2} = \bar{k}_{pq}(z). \qquad (3)$$

The boundary conditions along the taper are

$$E_z = -\frac{1}{2}\left(E_x \frac{da}{dz}\right), \qquad x = \pm\frac{a}{2};$$

$$E_z = -\frac{1}{2}\left(E_y \frac{db}{dz}\right), \qquad y = \pm\frac{b}{2}. \qquad (4)$$

Substituting (1) into Maxwell's equations in rectangular coordinates with due care exercised on differentiations and boundary conditions, and performing the integration over the cross section with appropriate normalization factors, we obtain the following telegraphist's equations for the three modes:

$$\frac{dI_{10}}{dz} = -\frac{\Gamma_{10}^2}{j\omega\mu}V_{10} + \left(\frac{1}{C_{10}}\frac{dC_{10}}{dz} + \frac{1}{2a}\frac{da}{dz}\right)I_{10}$$

$$\frac{dV_{10}}{dz} = -j\omega\mu I_{10} - \left(\frac{1}{C_{10}}\frac{dC_{10}}{dz} + \frac{1}{2a}\frac{da}{dz}\right)V_{10} + \frac{C_{12}}{C_{10}}\frac{1}{b}\frac{db}{dz}V_{12} + \frac{\bar{C}_{12}}{C_{10}}\frac{1}{b}\frac{db}{dz}\bar{V}_{12}$$

$$\frac{d\bar{I}_{12}}{dz} = -j\omega\epsilon\bar{V}_{12} - \frac{ab\bar{C}_{12}^2}{4}\left(\frac{b^2}{4a^2}\frac{1}{a}\frac{da}{dz} + \frac{1}{b}\frac{db}{dz}\right)\bar{I}_{12} - \frac{C_{10}\bar{C}_{12}}{2}a\frac{db}{dz}I_{10} + \frac{C_{12}\bar{C}_{12}}{2}\left(b\frac{da}{dz} - a\frac{db}{dz}\right)I_{12}$$

$$\frac{d\bar{V}_{12}}{dz} = -\frac{\bar{\Gamma}_{12}^2}{j\omega\epsilon}\bar{I}_{12} + \frac{ab\bar{C}_{12}^2}{4}\left(\frac{b^2}{4a^2}\frac{1}{a}\frac{da}{dz} + \frac{1}{b}\frac{db}{dz}\right)\bar{V}_{12}$$

$$\frac{dI_{12}}{dz} = -\frac{\Gamma_{12}^2}{j\omega\mu}V_{12} + \left(\frac{1}{C_{12}}\frac{dC_{12}}{dz} + \frac{1}{2a}\frac{da}{dz} - \frac{1}{2b}\frac{db}{dz}\right)I_{12} - \frac{C_{10}C_{12}}{2}a\frac{db}{dz}I_{10}$$

$$\frac{dV_{12}}{dz} = -j\omega\mu I_{12} - \left(\frac{1}{C_{12}}\frac{dC_{12}}{dz} + \frac{1}{2a}\frac{da}{dz} - \frac{1}{2b}\frac{db}{dz}\right)V_{12} - \frac{C_{12}\bar{C}_{12}}{2}\left(b\frac{da}{dz} - a\frac{db}{dz}\right)\bar{V}_{12} \qquad (5)$$

where

$$\Gamma_{1q}^2(z) = k_{1q}^2 - \omega^2\epsilon\mu \quad \text{and} \quad \bar{\Gamma}_{12}^2(z) = \bar{k}_{12}^2 - \omega^2\epsilon\mu. \qquad (6)$$

For gentle tapers reflections can be neglected and (5) can be simplified into three coupled first-order differential equation in forward-wave amplitudes A_{10}, $\bar{A}_{12}$, and A_{12} as:

$$\frac{dA_{10}}{dz} = -\Gamma_{10}(z)A_{10} + \zeta_{10\overline{12}}(z)\bar{A}_{12} + \zeta_{1012}(z)A_{12}$$

$$\frac{d\bar{A}_{12}}{dz} = -\bar{\Gamma}_{12}(z)\bar{A}_{12} - \tau_{10\overline{12}}(z)A_{10} + \zeta_{12\overline{12}}(z)A_{12}$$

$$\frac{dA_{12}}{dz} = -\Gamma_{12}(z)A_{12} - \zeta_{1012}(z)A_{10} - \zeta_{12\overline{12}}(z)\bar{A}_{12} \qquad (7)$$

where

$$\zeta_{10\overline{12}} = \frac{\sqrt{2}\,a\frac{db}{dz}}{b\sqrt{4a^2+b^2}}\left\{\left[1 - \left(\frac{\lambda_0}{2a}\right)^2\right] \cdot \left[1 - \frac{(4a^2+b^2)\lambda_0^2}{(2ab)^2}\right]\right\}^{1/4}$$

$$\zeta_{1012} = \frac{\frac{db}{dz}}{\sqrt{2}(4a^2+b^2)}\left[1 - \left(\frac{\lambda_0}{2a}\right)^2\right]^{1/4} \cdot \left[1 - \frac{(4a^2+b^2)\lambda_0^2}{(2ab)^2}\right]^{-1/4}$$

$$\zeta_{12\overline{12}} = \frac{2}{4a^2+b^2}\left[1 - \frac{(4a^2+b^2)\lambda_0^2}{(2ab)^2}\right]^{1/2} \cdot \left(b\frac{da}{dz} - a\frac{db}{dz}\right). \qquad (8)$$

From (5) and (8) we indeed show that in general $TE_{12}^{\square}$, $TM_{12}^{\square}$, and $TE_{12}^{\square}$ modes are mutually coupled in rectangular waveguide tapers. However, the second-order cross coupling between the $TM_{12}^{\square}$ and $TE_{12}^{\square}$ modes vanishes for square tapers or rectangular tapers with sides a and b changing at the same rate along the taper axis. For square tapers $\zeta_{12\overline{12}}=0$, (7) can be reduced to (22) in the paper by Tang[1] if the tapering of the guides is very gentle and the total couplings

$$\int_0^l \zeta_{10\overline{12}}dz \quad \text{and} \quad \int_0^l \zeta_{1012}dz$$

are all small. This is so because for weak interaction between all pairs it is often sufficient to consider only the coupling between two modes at a time.

Examination of coupling coefficients in (8) shows that at the cutoff of the converted modes $\zeta_{10\overline{12}}$ ($TE_{10}^{\square}$ to $TM_{12}^{\square}$) and $\zeta_{12\overline{12}}$ (between $TE_{12}^{\square}$ and $TM_{12}^{\square}$) have, respectively, a zero of the order $0^{1/4}$ and $0^{1/2}$, whereas ζ_{1012} ($TE_{10}^{\square}$ to $TE_{12}^{\square}$) has an infinity of the order $0^{-1/4}$. Accordingly, the overall coupling coefficient for square tapers in converting the $TE_{10}^{\square}$ mode into both $TM_{12}^{\square}$ and $TE_{12}^{\square}$ modes, simultaneously, also tends to approach at cutoff an infinity of the order $0^{-1/4}$ as the coupling coefficient for circular tapers does in converting the TE_{11}° mode into the TM_{11}° mode at cutoff frequency of the converted mode. In other words the physically reasonable conclusion is that the general overall behavior of waveguides is similar, whether the waveguide is rectangular, square, circular, or elliptical; i.e., the overall coupling at cutoff of a converted mode or modes approaches an infinity ($0^{-1/4}$).

Comparison of the two coupling coefficients $\zeta_{10\overline{12}}$ ($TM_{10}^{\square}$ to $TM_{12}^{\square}$) and ζ_{1012} ($TE_{10}^{\square}$ to $TE_{10}^{\square}$) shows that at frequencies far away from cutoff the magnitude of $\zeta_{10\overline{12}}$ is twice as large as ζ_{1012} for square tapers. Accordingly for relatively long square tapers which go through cutoff region, the total coupling $\int\zeta_{10\overline{12}}dz$ between the $TE_{10}^{\square}$ and $TM_{12}^{\square}$ modes is always larger than the total coupling $\int\zeta_{10\overline{12}}dz$ between the $TE_{10}^{\square}$ and $TE_{12}^{\square}$ modes in spite of the fact that at cutoff $\zeta_{10\overline{12}}\rightarrow 0^{1/4}$ and $\zeta_{1012}\rightarrow 0^{-1/4}$. Experimentally this is also verified by measuring and comparing the magnitudes of the

$TM_{11}^{\bigcirc}$ and $TE_{31}^{\bigcirc}$ modes in circular waveguide. Since we cannot separate the degeneracy of $TM_{12}^{\square}$ and $TE_{12}^{\square}$ modes in square waveguide measurements, the mode correspondence of $TM_{12}^{\square} \leftrightarrow TM_{11}^{\bigcirc}$ and $TE_{12}^{\square} \leftrightarrow TE_{31}^{\bigcirc}$ enables us to remove the degeneracy and to measure and show that $TM_{11}^{\bigcirc} > TE_{31}^{\bigcirc}$ in a circular waveguide by using a long straight, untapered, transition with a square of sides d at one end and a circle of diameter d at the other end.

The three-mode case (7) with $\zeta_{12\overline{12}}=0$ (square tapers or rectangular tapers with sides a and b changing at a same rate) can be solved by an iterative procedure under the initial boundary conditions:

$$A_{10}(0) = 1, \qquad \overline{A}_{12}(0) = 0, \quad \text{and} \quad A_{12}(0) = 0.$$

Without going into details, we write down the solutions:

$$A_{10}(l) = \exp\left[-\int_0^l \Gamma_{10} dz\right]\left[1 - \int_0^l \zeta_{10\overline{12}}(z) \exp\,[\overline{S}_z] \int_0^z \zeta_{10\overline{12}}(y) \exp\,[-\overline{S}_y] dy dz + \cdots \right.$$
$$\left. - \int_0^l \zeta_{1012}(z) \exp\,[S_z] \int_0^z \zeta_{1012}(y) \exp\,[-S_y] dy dz + \cdots \right]$$

$$\overline{A}_{12}(l) = -\exp\left[-\int_0^l \overline{\Gamma}_{12} dz\right] \int_0^l \zeta_{10\overline{12}}(z) \exp\,[-\overline{S}_z]\left[1 - \int_0^z \zeta_{10\overline{12}}(y) \exp\,[\overline{S}_y] \int_0^y \zeta_{10\overline{12}}(x) \exp\,[-\overline{S}_x] dx dy \right.$$
$$\left. - \int_0^z \zeta_{1012}(y) \exp\,[S_y] \int_0^y \zeta_{1012}(x) \exp\,[-S_x] dx dy + \cdots \right] dz$$

$$A_{12}(l) = -\exp\left[-\int_0^l \Gamma_{12} dz\right] \int_0^l \zeta_{1012}(z) \exp\,[-S_z]\left[1 - \int_0^z \zeta_{1012}(y) \exp\,[S_y] \int_0^y \zeta_{1012}(x) \exp\,[-S_x] dx dy \right.$$
$$\left. - \int_0^z \zeta_{10\overline{12}}(y) \exp\,[S_y] \int_0^y \zeta_{10\overline{12}}(x) \exp\,[-S_x] dx dy + \cdots \right] dz$$

where

$$S_i = \int_0^i (\Gamma_{10} - \Gamma_{12}) di,$$

$$\overline{S}_i = \int_0^i (\Gamma_{10} - \overline{\Gamma}_{12}) di, \qquad \text{with } i = z,\, y, \text{ or } x.$$

For gentle tapering the above solutions can be reduced to those of two-mode case obtained previously by others.[2,3]

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